# The Prime Knot Decomposition Theorem

Bachelor Thesis

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## Contents

# 1 Abstract

According to the theorem of Seifert in knot theory, every (oriented) knot can be written down as a "union" of finitely many oriented prime knots. The goal of this thesis is to understand this theorem and work very closely towards a rigorous proof of it. We will be looking at how a knot can be defined and treated exactly, and then go into another part of mathematics that is very important for the proof. Namely surfaces, since the proof of the theorem relies on the fact that every knot has an oriented surface of which it is the boundary, and specific properties of this surface. Most of all the genus, and how this genus changes when we take the union of surfaces. I will have to omit some proofs like the associativity of knot unions, but I will do as much as possible.

# 2 Introduction

I originally got pulled into the idea of doing my bachelor's thesis about knot theory because I have been in the scouts for a very long time. I saw the subject of knot theory so I made an appointment with my supervisor. Funnily enough any knots I know how to do from scouts are not even considered knots in knot theory. Because the ends of the rope are not connected you can just unknot it, and that would not be interesting mathematically. Take for example the reef knot or the bowline.

So me and my supervisor, M. Mueger, talked about what to do with the subject. We landed on the prime knot decomposition theorem because it is a more geometrical approach to knot theory which I was more interested in. I really like algebra, since we were considering covering the Alexander polynomial which can tell if knots are different, but I wanted to try something different.

Knot Theory's history is quite interesting, back in 1877 is when it started. P. G. Tait published some papers addressing enumeration of knots. Lord Kelvin made a theory of the atom that said that chemical properties of atoms are gained from an understanding of knots, due to how atoms can knot together. Tait would be motivated to start making the first knot tables.

First of all a knot can be seen as a piece of string with the ends attached, so we cannot unknot it in a trivial way. Tait's view on when two knots were equivalent was intuitive, they are equivalent if one could be deformed to appear as the other. This was insanely difficult, proving that two knots are the same can be a very hard task.

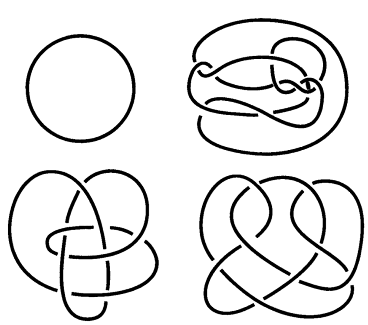


Figure 1: The top left is the trivial knot, the top right can be untangled, are the two knots from the bottom row the same? They seem to be non-trivial knots too, but can we be sure?

When Tait began working on this, the formal mathematics needed for many of the questions were not there yet. He had convincing arguments that his lists were complete, but they were not rigorous. Formally showing that two knots are distinct is still very hard to this day and it is why we have Alexander polynomials, and nowadays even better ones too. J. Alexander found them in 1928 using combinatoric arguments. If two knots have different polynomials then they are different. And he managed to prove that many of Tait's knots, that Tait thought were different, were in fact equivalent.
Not too long after Tait began researching this, topology became more firm in mathematics as a whole, and the tools to work on knot theory formally and exactly came to fruition. While using knots as representation for atoms became outdated quickly, mathematicians still found an interesting subject to work on and developed it.
Seifert made his big discovery in the early 1930's. If the knot is a boundary of a surface in 3-space, then we can study the surface to effectively study the knot. It gave efficient means to compute many of the known invariants of knots. And this thesis is going to focus mostly on the discoveries in this category.
I will start the thesis on how to define a knot properly of course. You might have some intuitive ideas but since we would like to remain in realistic ideas of a knot, we need to be careful. We have to avoid so called "wild knots" and make sure we do not accidentally create an uninteresting equivalence relation.

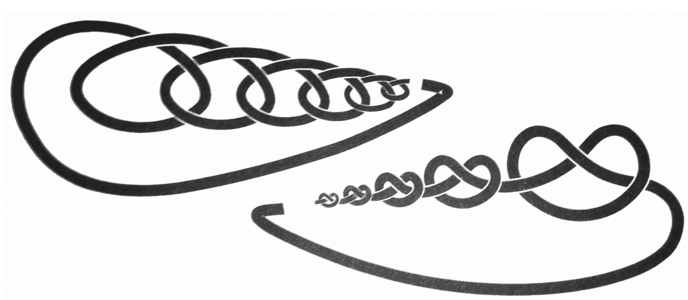


Figure 2: This is not realistic.

I will try to make it intuitive by often using a very simple example of a non trivial knot. One that will probably have crossed your mind thinking about one. The trefoil knot:

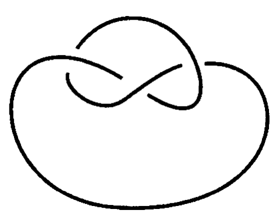


Figure 3: the trefoil knot

At first we may define our knots using simple curves in 3-space, but after this we will see why it is useful to treat knots exactly as we treat them here in a meta way. As drawings on paper that represent curves in a 3-space. This may sound funny but is integral for the sections coming hereafter. Since we will use the perspective we look at the knot, to construct

the surface mentioned. The one that Seifert uses. But first we use this idea to compose knots together.
Composing knots is very simple and intuitive, take a look back at the second wild knot above. That is just the trefoil knot composed an infinite number of times. However the properties of composing knots may be difficult. Where we cut either knots to attack them together may matter, when we have 4 ends to connect together, how do we choose to connect them, because this can be done in two ways? Sadly I cannot prove everything in this section but it gives us some interesting questions.
When we construct the surface that Seifert used later, we need to go into the details of how surfaces work. Since this is not common in the curriculum of the Radboud University, I will treat it as completely as possible. We will talk about how to construct the surface of which a specific knot is the boundary, and then speak about the surface's genus. The genus is essentially the number of torus-like holes a surface has.
With all of this combined we can get to our conclusion. Every knot can be uniquely split into a sequence of non trivial prime knots. A prime knot is very similar to the idea of a prime ideal from algebra. Trying to split a prime knot always ends up in having a union of a trivial knot, and the original knot. The proof of this prime decomposition is quite long and consists of working with surfaces. I hope you are looking forward to it.
Lastly I want to add that I took my time with this thesis. When the covid-19 pandemic hit in early 2020, I was having personal issues that made it harder to focus on university and caused me to delay handing in my thesis. The next year I started my master's degree, despite not having finished my bachelor's, and it was so difficult that I did not have time to work on my thesis at all. My personal issues only ended up getting worse and I even suffered a burnout. Throughout all of this, I had a lot of stress which often caused poor communication on my end. I just want to apologize to my supervisor, M. Mueger, for taking so long. And I hope this suffices.

## 3 How to define a knot

We want a knot to be based on our idea of a knotted loop in the real world. here is an intuitive idea that sadly fails to satisfy what we want it to be.

**Definition 3.1** A loop $\gamma : [0,1] \to \mathbb{R}^3$ is a **weak knot** if the following holds:

$$\gamma(x) = \gamma(y) \to (x = y \vee (x, y) \in \{(1,0),(0,1)\})$$

**Remark 3.2** At first this makes a lot of sense, it is just a loop in $\mathbb{R}^3$ that does not intersect itself. The reason that this may fail to be a knot is that it allows the possibility of wild knots. These wild knots go against our intuitive idea of a knotted loop, as they are impossible to recreate in the real world.

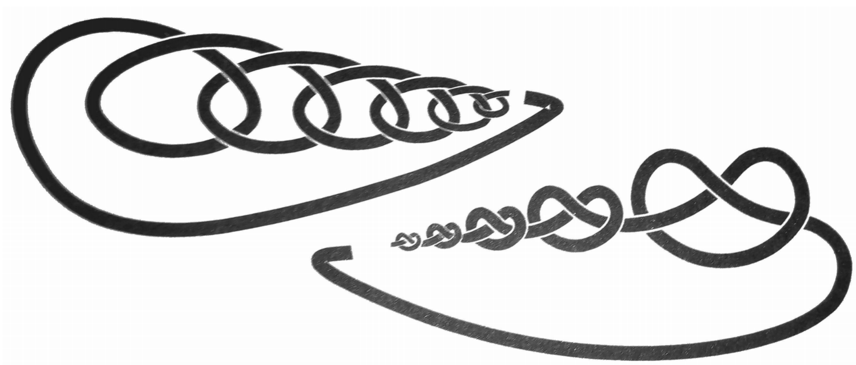


Figure 4: Two wild knots

Instead we will avoid wild knots by defining a knot using a finite number of ordered points in $\mathbb{R}^3$ and connecting them with linear paths. This is also called a **closed polygonal curve**.

**Definition 3.3** A loop $K : [0,1] \to \mathbb{R}^3$ is a **knot** if the following holds:
i) $K$ is a weak knot.
ii) There exists an $n \geq 3$, and a finite ordered set $(k_1, \dots, k_n)$ such that $Ran(\gamma) = [k_1, k_2] \cup [k_2, k_2] \cup \cdots \cup [k_{n-1}, k_n] \cup [k_n, k_1]$. Where $[k_i, k_{i+1}] = \{t \cdot k_i + (1-t) \cdot k_{i+1} \mid t \in [0,1]\}$.

Let us also give the points in the finite ordered set a name.

**Definition 3.4** If a finite ordered set $(k_1, \dots, k_n)$ defines a knot, and for any $i \in \{1, \dots, n\}$ $(k_1, \dots, k_n) \backslash \{k_i\}$ defines a different knot. Then the points $\{k_1, \dots, k_n\}$ are called the **vertices** of a knot

While this definition will not allow our knots to be smooth, we can still smoothly represent them. Also the fact that smoothly drawn non-wild knots can be approximated with a finite number of points is intuitively clear.

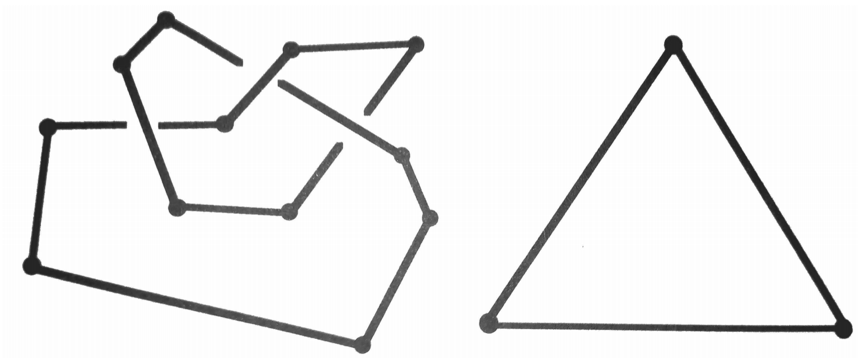


Figure 5: A trefoil knot and the unknot that are not smoothly represented

**Remark 3.5** Before we continue I want to emphasize the specific concept of "is intuitively clear". Because this is a very dangerous phrase to use in mathematics. Often enough the phrase would indicate that there is significant hand waving in a proof. I would like to explain why it is less dangerous to use here because we are working with polygonal curves using an example.
An important theorem that seems intuitive, but is in actuality hard to prove, is the Jordan curve theorem. A Jordan Curve is an injective map $\gamma : S^1 \to \mathbb{C}$. Simply speaking it is a loop on a plane that does not intersect itself. The Jordan curve theorem states that any Jordan curve divides the complex plane into exactly two regions, of which it is the common boundary. To prove this theorem is actually quite difficult in general due to the existence of curves like the Peano curve, a famous space filling curve. It may not be injective but it shows that continuous paths can have unexpected, counterintuitive topological properties, which are not shared by "nice" paths like polygonal or differentiable ones. At that point one might worry that the Jordan curve could have a hidden caveat. Luckily the proof is very simple when assumed that the curve is polygonal. A proof of the Jordan Curve theorem can be found in Chapter 4 of Winding Around [7]. And it includes a step by step exercise where you prove the polygonal curve version, which is much simpler to work with since it represents our intuitive idea of curves.
In knot theory rigorous and fully worked out proofs may become very long, but since we are working with polygonal curves the argument of "is intuitively clear" is used and accepted more often.

Going back to our definition of a knot, this still has problems. We can represent the unknot with both $((0,0,0),(1,1,0),(-1,1,0))$ and $((1,1,1),(2,2,1),(0,2,1))$. We can also pick a random $i \in \mathbb{N}$ and redefine a knot $(k_1,\ldots,k_n)$ as $(k_{1+i \, mod \, n},\ldots,k_{n+i \, mod \, n})$ and all we would have done is pick a different starting point. But we can also remove points in some cases, like the trivial case where some point $p_j$ is on the line segment $[p_{j-1},p_{j+1}]$. Do note that in this case the point $p_j$ is not a vertex of the knot.
To fix this issue, we need to define an equivalence relation so we can talk about equivalence classes of knots instead. This way we can wrap, unwrap, and stretch knots more similar to our intuitive idea of a knot. We want something like isotopy, but more specific to closed polygonal curves.

**Definition 3.6** A knot is called an **elementary deformation** of another knot, if one of the two knots is determined by $(k_1, \ldots, k_n)$ and the other by $(k_0, k_1, \ldots, k_n)$ where:
(1) $k_0 \notin [k_n, k_1]$
(2) The triangle spanned by $k_0$, $k_1$, and $k_n$ only intersects the smaller knot in $[k_1, k_n]$ or more formally: $\{k_0 + t \cdot k_1 + u \cdot k_n \mid t, u \in [0, 1]\} \cap ([k_1, k_2] \cup \cdots \cup [k_{n-1}, k_n] \cup [k_n, k_1]) = [k_1, k_n]$

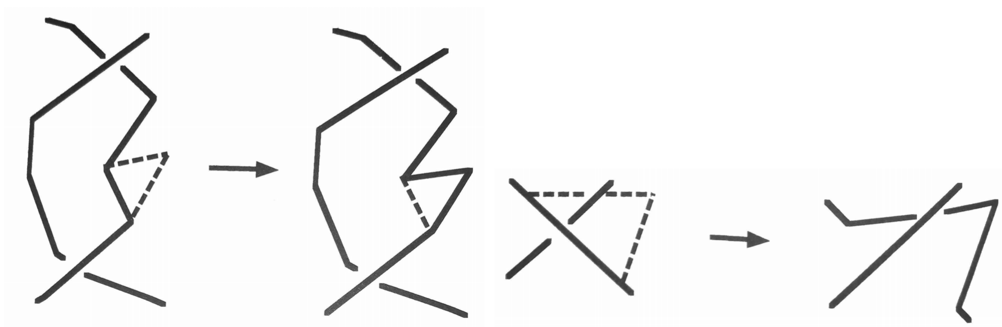


Figure 6: A legal move on the left and an illegal move on the right

**Example 3.7** What this allows us to do is add a point to a knot, or remove one, while making sure any step does not allow for the knot to pass through itself.

**Definition 3.8** Two knots are equivalent if we can turn one into the other using only finitely many elementary deformations. That this is indeed an equivalence relation, is trivial.

**Remark 3.9** We want knots to only be deformed into each other in a finite number of steps for a good reason. Because allowing an infinite number of elementary deformations will make all knots equivalent to the unknot. This can easily be seen using the following diagram:

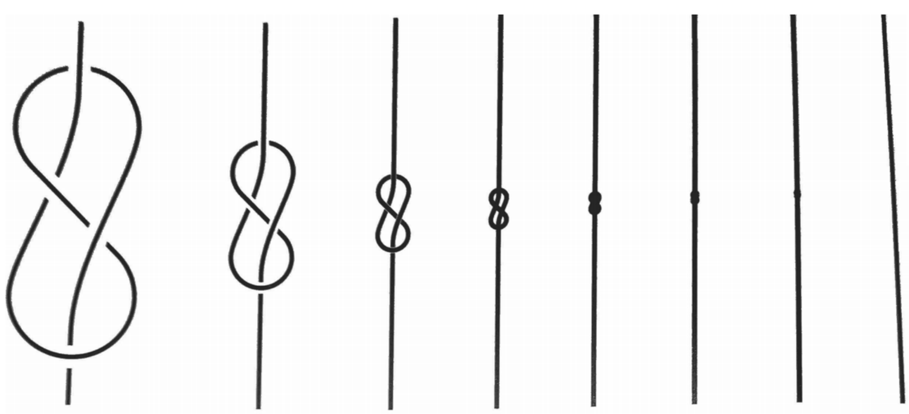


Figure 7: Shrinking a simple knot into a single point using an infinite number of elementary deformations

**Terminology**:
-Two knots that are equivalent are also called **similar**, or just the same.
-Two knots that have different equivalent classes are called **distinct**.
-A knot that is similar to the trivial knot is called **unknotted**.

From now on we will only be talking about equivalence classes of knots, but referring to them as knots. There is no harm in blurring the line between a knot and its equivalence class.

# 4 Knot diagrams

In the previous sections, I have mostly shown examples of knots by drawing them. It is impossible to truly draw a curve in $\mathbb{R}^3$, because it is a three-dimensional object and drawings are 2 dimensional. What we have been using are essentially projections of the curve, with some extra information. While this might seem strange to point out, we are going to use these projections to define some properties and obtain some necessary results.

Now projections by their nature are more often than not, not injective. If a knot has even one crossing we lose injectivity, that makes it so only the trivial projections of the unknot are injective. So while knot projections will almost never be injective, we still want to make sure that the entire knot is visible on this two-dimensional plane.

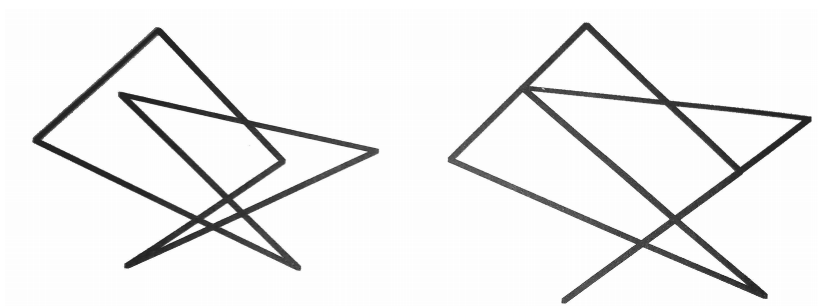


Figure 8: Two projections of the same knot

To fix this issue we introduce another definition.

**Definition 4.1** A knot projection is called a **regular projection** if no three points on the knot project to the same point on the plane, and no vertex projects to the same point as any other point on the knot.

By proving that every knot has a regular projection, and formally defining how to draw a knot to give enough information about its shape in 3-space. We can treat a knot and a its drawing as the same object which will be vital for upcoming definitions and proofs. That is the goal of this section, and this follows from the following three results.

First result: If a knot does not have a regular projection, we can find an equivalent knot that is only slightly different, that does have one. This idea of slight difference, is formalized by using the distance between vertices.

**Theorem 4.2** *Let $K$ be a knot determined by $(k_1, \dots, k_n)$. For every $\varepsilon > 0$ there is a knot $J$ defined by $(j_1, \dots, j_n)$ such that:*
*-$\max\{d(k_i, j_i) \mid i \in \{1, \dots, n\}\} < \varepsilon$ (this is slight difference)*
*-$K$ and $J$ are similar.*
*-$J$ has a regular projection*

**Proof:** Without loss of generality, let us assume that we are projecting our knots on the XY plane. If the projection of $K$ is regular, the result is trivial so we will assume it is not a regular projection.

We can solve the knot not having a regular projection on the XY plane in a pretty simple way. We make sure that two things are not the case.
1) Two vertices cannot be projected on the same point (Two vertices cannot have the same X and Y coordinates).
2) Two edges cannot overlap, we fix this by making sure no three vertices are on one line (three vertices cannot be on the same plane if that plane is perpendicular to the XY plane)
First find an $\varepsilon_i > 0$ for every $i \in \{1, \ldots, n\}$ such that for every vertex $k_i$ we have an open ball of radius $\varepsilon_i$ with the following property: if we move $k_i$ somewhere else in this open ball, the new knot is still equivalent to $K$.
That this is possible, is easily proven because we are working with a finite number of points. If we deny this possibility, we can see that $K$ cannot be a knot. Causing a contradiction.
If a pair of two vertices have the same X and Y coordinates, move one of them inside their open ball. Do this until no two vertices are projected onto the same point. This is possible for any open ball of radius $\varepsilon \leq \varepsilon_i$. This covers 1).
Now we construct a plane for every pair of vertices, perpendicular to the XY plane. If any vertex is in a plane that is not generated by itself, move it somewhere in its open ball making sure it does not intersect any other plane. This is obviously possible for any open ball of radius $\varepsilon \leq \varepsilon_i$ since we only have a finite number of planes. This covers 2).
And thus we have constructed our knot $J$ for any $\varepsilon > 0$. □

Second result: If a knot has a regular projection, we can find infinite equivalent knots, that are only slightly different, that also have a regular projection.

**Theorem 4.3** *Let $K$ be a knot determined by $(k_1, \ldots, k_n)$ that has a regular projection. There exists an $\varepsilon > 0$ such that for any knot $J$ determined by $(j_1, \ldots, j_n)$:*
$\max\{d(k_i, j_i) \mid i \in \{1, \ldots, n\}\} < \varepsilon \to K$ *is similar to $J$ and $J$ has a regular projection.*

**Proof:** We start similar to the last theorem. Without loss of generality assume that we are projecting to the XY plane. We again find a radius for each vertex, so the open ball around each surface makes sure moving the vertex around keeps the knot similar. And again we construct the planes for every pair of points, this time our knot already has a regular projection so we know that every plane only intersects the two points that it was constructed with.
Now we take $\varepsilon = \min\{\varepsilon_i \mid i \in \{1, \ldots, n\}\}$ and make all the open balls this radius. Now we check if any of the balls intersects with one of the planes. If it does not, we are done. If it does, take $0.5 \cdot \varepsilon$ and check it again. Within a finite number of steps we have the $\varepsilon$ we were searching for. □

**Definition 4.4** A **knot diagram** is a regular projection of a knot on a two-dimensional plane, that is also showing which parts of the knot pass under other parts using over and under crossings. These are represented by leaving out a small part of the curve that passes under another part.

Figure 9: A knot diagram of a non trivial knot

The following third result confirms that a knot diagram gives us enough information, that we can obtain only one equivalence class of knots from it.
After this we can interchangeably talk about a knot as either an equivalence class of curves in $\mathbb{R}^3$, or a diagram. We can start talking about edges and crossings like they belong to the knot itself.

**Theorem 4.5** *If two knots, $K$ and $J$ have regular projections and identical knot diagrams, then they are equivalent knots.*

**Proof:** Without loss of generality we are projecting to the XY plane, $K$ is defined by $(k_1, \dots, k_n)$ and $J$ by $(j_1, \dots, j_n)$ where each pair of $(k_i, j_i)$ are the vertices that are projected to the same point, and thus differ only in the Z coordinate. When the diagram indicates an over crossing, we will assume that the Z coordinate of the line segment is bigger than the Z coordinate of the under crossing line segment.
We are going to transform $K$ into $J$ with a simple algorithm. First we cherry pick all the vertices whose two neighbouring edges do not have a crossing on the projection. We can freely change the Z coordinates without worrying about it. Now we handle the crossings...
We are going to transform $K$ into $J$ with a simple algorithm. First we mark all crossings on the projection, and for each crossing we take an $\varepsilon > 0$ so in the closed circle around the crossing, there are no other crossings and no other vertices. This way the boundary of all the closed circles intersect the knot projection in exactly four points. We add these as extra vertices for both of our knots. This means that our original vertices now have the following property: The two neighbouring edges of each $k_i$ and $j_i$ do not have a crossing on the projection. Because of this we can freely change the Z coordinate of each $k_i$ without changing the knot. Thus we change them to match the Z coordinate of the respective $j_i$, and now we only have to move the new points we added around each crossing.
For each four vertices that form a crossing, do the following steps. We first pick the two vertices that correspond to the over crossing on the diagram. If these need to be moved up in the Z-axis towards their position in $J$, then we do this. If these need to be moved down we may do this unless it has to cross the line piece that corresponds to the under crossing. In that case we move the vertices from the under crossing to its position in $J$ instead. It is guaranteed that these can be moved downwards into their desired position. This guarantee follows from the fact that $K$ and $J$ have an identical knot diagram. After one of these vertex pairs are moved into their position in $J$ this means we can just move the other pair of vertices to their position in $J$ without issue. □

## 5 Composition of knots

Before we start composing knots, I need to define what an oriented knot is.

**Definition 5.1** An **oriented knot**, is a knot with an ordering of its vertices. The ordering must be chosen, so it describes the original knot. Two orderings of a knot are equivalent if they differ by a cyclic permutation.

**Example 5.2** Informally this just means you pick a direction to walk along the curve of the knot. And we represent this by noting a direction on the diagram of a knot.

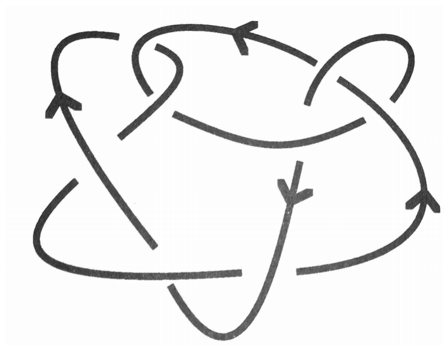


Figure 10: An oriented knot

To define the composition of knots, we need Theorem 4.5 as it is defined using knot diagrams.

**Definition 5.3** Let $K$ and $J$ be two oriented knots. For both knots, take an edge on the outside of the knot diagram, and pick a point on this edge. For a small $\varepsilon > 0$ we make a disk of radius epsilon around the chosen points, we take epsilon small enough so the disk does not overlap with any other edges or crossings, we remove everything in those disks. We have created 4 end points on the two knots: $(k_1', k_2')$ and $(j_1', j_2')$. Where in the orientation of the original knots is from $k_1'$ to $k_2'$ and from $j_1'$ to $j_2'$. We now add two extra edges, that we make sure don't create any extra crossings, from $k_1'$ to $j_2'$ and from $k_2'$ to $j_1'$. This preserves the orientation on both knots. Using this we have made a new knot, called the **connected sum** $K\#J$.

**Example 5.4** -This shows how to compose two knots. You just have to make sure that orientation is preserved.

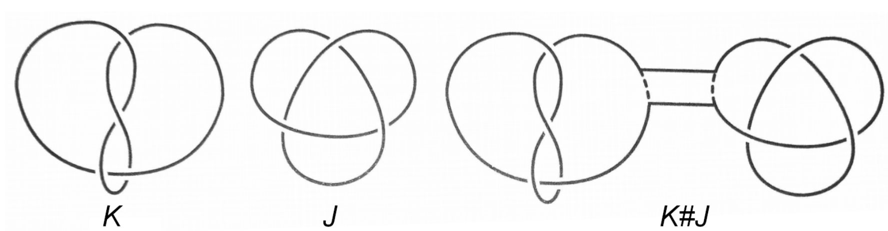



Figure 11: Composing two knots the way the definition described

-The easiest way to think about composing knots, is with boxed knots.

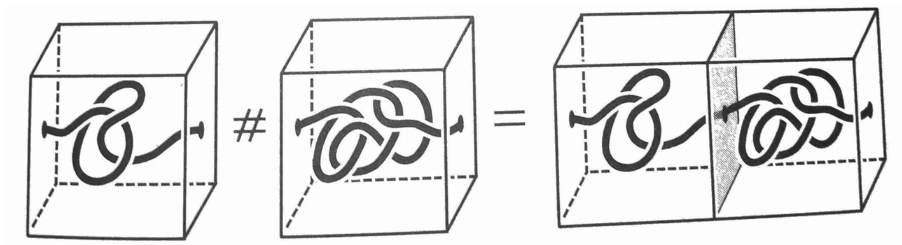



Figure 12: Composing knots with the idea of boxed knots. The end points will be strung together without adding another knot.

Looking at composing knots like boxed knots, it is easier to see some facts. That the composition of two oriented knots always produces the same knot is clear if you assume that it does not matter where you cut the knot. You just move the entirety of the knot into the box. Seeing why it does not matter where we cut is the harder part to understand. More formally: a knot $K$ similar to $K'$ implies that for any knot $L$ we have $K\#L$ is similar to $K'\#L$.
The formal proof for this is long however and uses homeomorphisms on surfaces of which the knots are the boundary. Proving that $(K\#L)\#H$ is similar to $K\#(L\#H)$ is also surprisingly difficult despite how intuitive it looks here. The homeomorphisms needs to be made with great care and the proofs will be omitted from this text. Lastly it is obvious that composing a knot with the unknot changes nothing.
An extra thing is this. If we compose two oriented knots, but instead connect them so the orientations do not match. We can indeed produce a different knot. Ignoring orientation when composing knots, leads to having two possibilities for composition. But we will focus on orientation preserving composition.
Lastly, we may note that knot composition is commutative. To swap the order of two knots is very easily imagined with boxed knots. Shrink one box to be very small, walk along the curve of the other knot following the orientation, and make it big once the small box left the bigger box.

Composing knots also brings up the idea, of analyzing how knots can be split. This brings us to the main subject of this paper. First, do non trivial knots have an inverse?

**Proposition 5.5** *Let $K$ be an oriented knot that is non-trivial. There does not exist an oriented knot $J$ such that $K\#J$ is unknotted.*

We cannot prove this proposition just yet, so for now please assume that it holds. Assuming that no inverses exist, we can talk about prime knots, our end goal.

**Definition 5.6** A **prime knot**, is an oriented knot $K$ such that for any knot composition $J_1\#J_2$ that is equivalent to $K$: $J_1$ is unknotted or $J_2$ is unknotted.

**Remark 5.7** If a non-trivial knot $K$ existed with an inverse $K^{-1}$, prime knots would not be interesting. As we could say a lot of knots are not prime by using the following argument: Take a knot $J$, it is equivalent to $J\#K\#K^{-1} = (J\#K)\#K^{-1}$. We know that $K^{-1}$ is not trivial, so $J$ can only be prime if $J\#K$ is unknotted. Thus any knot can only be a prime knot if it is an inverse of every invertible knot. This would also make the unknot, not a prime knot.

Any non-trivial knot lacking an inverse, combined with the idea of prime knots might make you want to ask if every knot can be split into a composition of prime knots. Just like how every natural number $> 1$ can be written as a unique product of prime numbers. This is the case, and will be the result we are going to work towards.

**Theorem 5.8** *The prime knot decomposition theorem:*
*Every non-trivial oriented knot can be decomposed as the connected sum of nontrivial prime knots. And if $K = K_1\#\ldots\#K_n$ and $K = K'_1\#\ldots\#K'_m$ where each $K_i$, $K'_i$ are non-trivial prime knots, then $n = m$ and there exists a bijective function $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$ with the following property: $K_i$ is equivalent to $K'_{\sigma(i)}$ for $i \in \{1, \ldots, n\}$.*

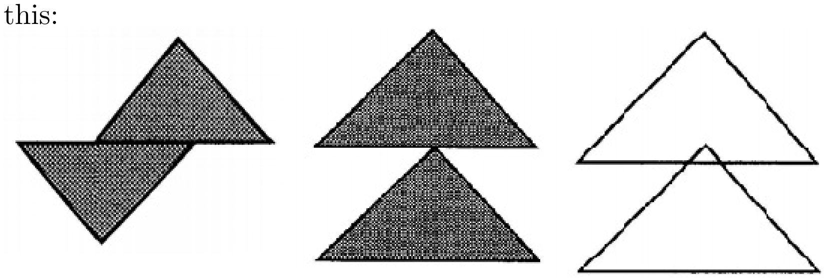
But to achieve this, we need to jump into the subject of surfaces. Because the proof of this theorem requires a very specific result, that for every oriented knot we can find an oriented surface such that its boundary is the knot.

# 6 Surfaces

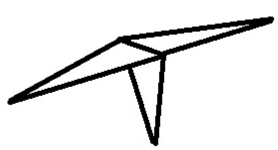
We ended up defining a knot using a finite ordered set of points. And we are going to do something similar for the surfaces we are going to work with. But our building blocks in this case are triangles of the following form:
Take three points in $\mathbb{R}^3$ that are not in a single line, let us call them $t_1, t_2$, and $t_3$, then our triangle surface is the following set of points.
$\{x \cdot t_1 + y \cdot t_2 + z \cdot t_3 \mid x + y + z = 1, \text{ and } 0 \leq x, y, z \leq 1\}$

**Definition 6.1** We define a **polyhedral surface** as a finite collection of these triangle surfaces with the following properties: (from Gilbert-Porter p56)
-The intersection of two triangles is either a shared vertex, a shared edge, or empty.
-Any edge, is an edge of either one or two triangles.
-For a vertex $t$, write $St(t)$ for the union of all triangles that have $t$ as a vertex, also called the star of $t$. For all vertices $t$ of the surface, $St(t)$ is homeomorphic to a disc.

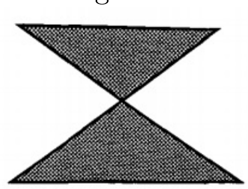
**Example 6.2** Here is a short explanation of each of the rules:
-The first condition emphasizes that triangles can only meet in a 'nice' manner. So not like this:



-The second condition just disallows T shapes like this:



-This condition is to eliminate possibilities of a surface like 2 triangles meeting only at one vertex, since the star of this vertex would be homeomorphic to 2 discs that touch at a point, not a single disc.



The way we defined surfaces by using triangles might make you think about how 3D objects are made in computer programmes and video games. Those objects are made using small triangles called polygons, and just using enough of them allows you to make smooth looking surfaces, while still only using a finite number of polygons.

Just like with knots, polyhedral surfaces can be drawn smoothly, and it is intuitive that smooth surfaces can be closely approximated using a polyhedral surface, as long as they are not wild.

We need some important extra definitions for surfaces, to get to our desired result. We start with the boundary of a surface, which is very intuitive. And then move on to a surface' orientation which needs a little more explanation. Lastly we talk about homeomorphism and that is where we can start obtaining some propositions and theorems.

**Definition 6.3** Given a surface, the union of all the edges that are only an edge of one triangle, is called the **boundary** of the surface. And the connected components of the boundary are called the **boundary components**.

**Definition 6.4** A polyhedral surface is called **orientable** if it is possible to orient the boundary of each of the triangles in such a way that when two triangles meet along an edge, those triangles are oriented along that edge in opposite directions.

**Example 6.5** The intuitive way to think about this, is that an orientable surface has two sides. A disk has a top and a bottom, a ball has an outside and an inside, and similarly for a ring. The other way to think about it is that if you were to live on this surface, you cannot go to the other side of the surface without jumping into a hole of its boundary. So if the surface does not have a boundary it is impossible. two famous examples of non orientable surfaces are the Möbius strip and the Klein bottle.

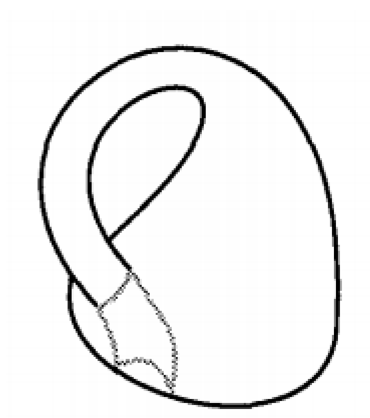
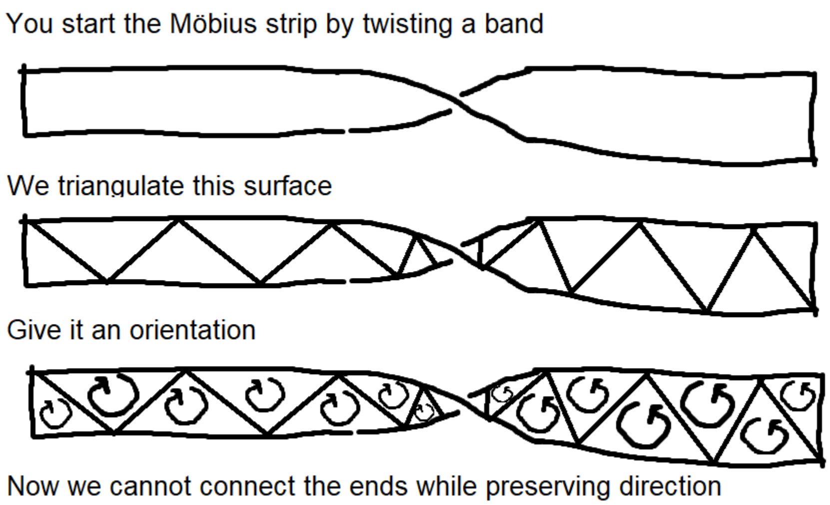


Figure 13: The Klein bottle, you cannot distinguish an inside and outside of this bottle.

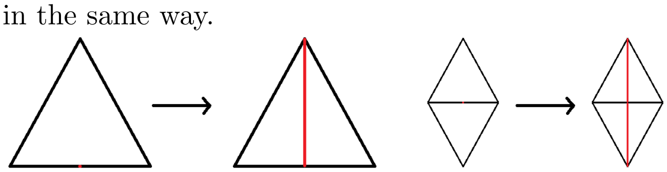
I will now attempt to give Möbius strip an orientation to show that it is indeed not orientable.



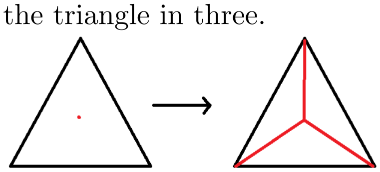

We already know when two surfaces are homeomorphic, from topology. We just need a continuous bijection from one surface to the other, with a continuous inverse. Do remember that homeomorphic is not the same as being able to continuously deform one surface into another without it intersecting itself in the process.
Take for example any knot, but instead of a curve, make it a small tube. This would make it homeomorphic to the torus, since it is just a tube with the ends glued together. If the knot is non-trivial it could not be continuously deformed into a torus without intersecting itself. Because, by definition of a non-trivial knot, we cannot do elementary deformations to change it into the unknot.

For polyhedral surfaces, the definition of when two surfaces are homeomorphic is a little more specific. First we need to be able to refine a polyhedral surface.

**Definition 6.6** We can refine a polyhedral surface by subdividing its triangles into more triangles. For each step you choose a triangle and subdivide it. After a finite number of steps you end up with the exact same surface but described using more triangles, this is called a **refinement** of the surface. Here are our options on subdividing a triangle:
-Choose a point on one of its edges but not on its vertices, and add one edge to split the triangle in two. If the edge you chose meets another triangle, that triangle needs to be split in the same way.



-Choose a point on the interior of the triangle, and add three edges to each vertex and split the triangle in three.



**Definition 6.7** Two polyhedral surfaces are **homeomorphic**, if we can take a refinement of the coarser surface such that the refined surface and the finer one have an equal number of vertices. And if we can then make a bijection from the vertices of one to the other, which has the following property: when three vertices in one surface bind a triangle, the corresponding three vertices on the other surface also bind a triangle.

**Proposition 6.8** *Take two Polyhedral surfaces $K$ and $J$, that are homeomorphic as described above. Then they are homeomorphic in the classic sense.*

**Proof:** Without loss of generality, assume we need to refine $K$, and let $K'$ be the refinement. Also define $v(K)$ as the set of vertices of the surface $K$. Our assumption tells us there exists a bijective function $f : v(K') \to v(J)$ with the property that if $(t_1, t_2, t_2)$ is a triangle in $K'$, then $(f(t_1), f(t_2), f(t_3))$ is a triangle in $J$.
For any $p \in K'$ let its triangle be $(t_{p_1}, t_{p_2}, t_{p_3})$ and $p = x \cdot t_{p_1} + y \cdot t_{p_2} + z \cdot t_{p_3}$. Define $f' : K' \to J$ by $f'(p) = x \cdot f(t_{p_1}) + y \cdot f(t_{p_2}) + z \cdot f(t_{p_3})$. We need only show that $f'$ is continuous since its inverse is constructed in the exact same manner.
-If $p$ is in the interior of its triangle, $f'$ is clearly continuous.
-If $p$ is on an edge/vertex of the triangle, and this edge/vertex does not meet another triangle, $f'$ is clearly continuous.
-If $p$ is on an edge/vertex of the triangle, and this edge/vertex meets another triangle, showing that it is continuous is an easy exercise for the reader. (First you need to show that if two triangles meet in $K$ then their corresponding triangles meet in $J$ from that point on it is pretty simple.) □

**Remark 6.9** Any oriented surface (which does not necessarily need to be polyhedral) has a **genus** which is an integer value that represents the number of holes that a surface has. Not in its boundary but in the surface itself. The torus is a great example of a surface with a genus of 1. You can interpret the hole in the surface like this: assume you are a creature living on the surface, you can walk around on the outside (and only the outside because it is also orientable). But you cannot fall in the hole, unlike if it were to have a hole in the surface.

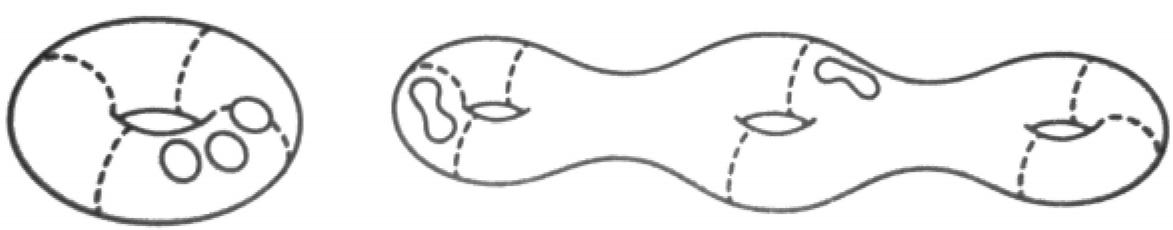


Figure 14: A surface with a genus of 1 on the left, and one with a genus of 3 on the right.

A ball with a puncture in its surface would have a genus of 0. That is because stretching out the hole, and flattening the ball would just make it a disc. But it is also in reverse, take a disc with a hole in it and you can stretch and mold it into a ball with two punctured points. So that obviously also has a genus of 0. Puncturing a torus thus also does not change its genus, but when given a random surface, finding its genus may prove difficult. If all you have to work with, is comparing it punctured tori.

The genus is a great way to classify a surface. I wanted to give you an intuitive idea of what a genus is, since the formal definition of the genus of a polyhedral surface does not give this intuitive idea. We will later on see that knots have a genus too. Which is going to be absolutely vital to proving the prime knot decomposition theorem.

We are going to start with the Euler characteristic because this is a simple invariant on polyhedral surfaces that can be used to calculate the genus of the surface.

**Definition 6.10** Take a polyhedral surface $K$ consisting of $T$ triangles, $E$ edges, and $V$ vertices. Then its **Euler characteristic** is $\chi(K) = T - E + V$

**Proposition 6.11** *The Euler characteristic does not change when a surface is refined.*

**Proof:** Because a refinement is a finite amount of steps, of single triangle subdivisions. We only have to check these three possible steps.
(i) When a triangle is split in two by adding a new vertex on an edge, and the edge we split does not meet another triangle: we gain +1 triangle, +2 edges, +1 vertex. $T+1-(E+2)+V+1 = T - E + V$
(ii) When the edge we split meets another triangle, we split two triangles in two: we gain +2 triangles, +3 edges, +1 vertex. $T + 2 - (E + 3) + V + 1 = T - E + V$
(iii) When we add a new vertex on the interior of a triangle: we gain +2 triangles, +3 edges, +1 vertex. Which is the same situation as (ii). □

**Corollary 6.12** *Let $K$ and $J$ be homeomorphic polyhedral surfaces. Then $\chi(K) = \chi(J)$*

Using this we can properly define the genus of a polyhedral surface.

**Definition 6.13** For a connected orientable surface $K$, where $B$ is its number of boundary components. The **genus** of $K$ is given by:

$$g(K) = \frac{2 - \chi(K) - B}{2}$$

**Theorem 6.14** *If two surfaces intersect in a collection of arcs contained in their boundary, the Euler characteristic of the union is the sum of their individual Euler characteristics minus the number of arcs of intersection.*

**Proof:** Suppose that each arc of intersection is a single edge of a triangle on each surface. Then the triangulations of the surfaces piece together to give a triangulation of the union. The count that is used to compute the Euler characteristic of each surface separately gets a contribution of 1 from each edge of intersections (-1 for the edge, and +2 for its endpoints.) Hence for the sum of the two Euler characteristics there is a contribution of +2 from each edge of intersections. However, in the union there is a contribution of only +1 from each edge. The result follows.
If each arc is not a single edge of a triangle, it can be arranged to be the union of edges, after subdividing. Again it turns out that the contribution of each arc toward the total Euler characteristic is +1, and the rest of the argument is the same □

**Corollary 6.15** *If two connected orientable surfaces intersect in a single arc contained in each of their boundaries, the genus of the union of the two surfaces is the sum of the genera of each.*

**Proof:** Take surfaces $K$ and $L$, note that one boundary component is lost in forming the union because it is orientable. We use Theorem 6.14 and the Euler characteristic of our union is $\chi(K) + \chi(L) - 1$. The Genus of the union is:

$$\begin{aligned} g(K \cup L) &= \frac{1}{2}(2 - (\chi(K) + \chi(L) - 1) - (B_K + B_L - 1)) \\ &= \frac{1}{2}(4 - \chi(K) + \chi(L) - B_K - B_L) \\ &= \frac{1}{2}(2 - \chi(K) - B_K) + \frac{1}{2}(2 - \chi(L) - B_L) \\ &= g(K) + g(L) \end{aligned}$$

□

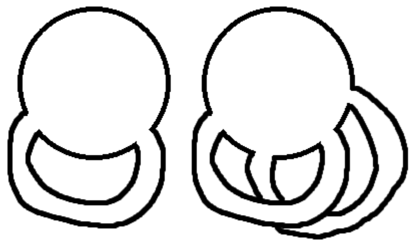
**Theorem 6.16** *If a connected orientable surface is formed by attaching bands to a collection of disks then the genus of the resulting surface is given by:*

$$\frac{2 - \#disks + \#bands - \#boundary\_components}{2}$$

**Proof:** We need only prove that the Euler characteristic is the $\#disks - \#bands$. We do not twist a band because we want the surface to stay orientable. Since a refinement does not change the Euler characteristic we can work with very coarse objects.
We represent a disk with a single triangle, 3 vertices, 3 edges, and 1 face give it a characteristic of 1. Connecting 2 disks with a band is possible with only 2 triangles. But it would share 2 edges and all vertices with the 2 disks, so it would add 2 faces and 3 edges but no vertices meaning that adding a band has a characteristic of -1. This also holds for adding a band that connects back to the same disk.
If we want to add an additional disk, it needs to be connected to the rest of the surface so it comes together with a band. We use Theorem 6.14, adding a new disk would not change the Euler characteristic, because it has characteristic 1 but the band gives an arc of intersection. With all of this, the formula holds. □

**Proposition 6.17** *The genus of a connected orientable surface which is formed by attaching bands to a collection of disks is* $\geq 0$

**Proof:** We want to minimize the formula seen in Theorem 6.16. Take $n \in \mathbb{N}$ disks, to make the surface connected we need $n - 1$ bands to connect them all together. The surface we have now might remind you of a tree graph, if the disks were vertices and the bands were edges, so you can see that we only have 1 boundary component and thus a genus of 0.
The idea is to maximize the amount of boundary components, while adding as few bands as possible. But you can intuitively see that adding one band, will add one boundary component. Afterwards we may add another band, but the amount of boundary components changes with either 1 or -1 each time. Because you only have a couple of options:
-You can connect two disks with a band, while making sure you keep orientation intact. This adds one boundary component.
-You can choose one disk and put both ends of the band on it. This action can either add or remove one boundary component depending how many bands are already attached and how.
Example:



The left surface has two boundary components, while the right surface has one. Basically, the first band split the boundary into two components, and using the second one you bind them back together. The conclusion is that it is not possible to have the number boundary components increase faster than the number of bands. Thus the genus cannot be negative. □

**Theorem 6.18** *If two surfaces intersect in a collection of circles contained in the boundary of each, the Euler characteristic of their union is the sum of their Euler characteristics.*

**Proof:** The argument is similar to that of Theorem 6.14. In computing the Euler characteristic of a surface, each boundary component contains an equal number of edges and vertices of the triangulation. Hence it contributes 0 to the total Euler characteristic. The same holds for the union. □

**Theorem 6.19** *Every connected surface with boundary is homeomorphic to a surface constructed by attaching bands to a disk.*

**Proof:** Here is a sketch of the proof: Fix a triangulation of the surface. A small neighbourhood of each vertex forms a disk. Thin neighbourhoods of the edges form bands joining the disk together. Hence a neighbourhood of the edges is homeomorphic to a union of disks with bands added. Two steps remain. The more difficult one shows that adding the faces has the same effect as not attaching certain of the bands. The other argument follows:
If a surface consists of two disks with a single band joining them, it is homeomorphic to a single disk with no bands attacked. We use this to transform any surface into a surface with only one disk, so pick any connected surface made of disks and bands. Take two disks that have at least one band between them. We pick one of those band and merge the disks removing the band, any extra bands in between them turn into bands from the new disk to itself. Any bands from other disks to any of these two disks become bands to this new disk. Because the surface is connected we can keep doing this until we end up with only one disk.□

This observation may not seem important at first, because the surface we are going to construct is going to have multiple disks. So that our knot may be its boundary. However doing calculations on knot invariants is easier if we can describe the surface with only one disk.
With this result we can also conclude that every connected oriented surface with boundary has a genus $\geq 0$. By applying a simple induction proof on the number of bands, we can also show that the genus is an integer in a very similar fashion of the proof of Proposition 6.17. This will also be important later.

**Proposition 6.20** *The genus of an orientable surface is an integer.*

**Lemma 6.21** *Two surfaces of the form: a disk with bands attached, are homeomorphic if and only if these conditions are met:*

*(i) They have the same number of bands.*
*(ii) They have the same number of boundary components.*
*(iii) both are either orientable, or non-orientable.*

Our goal of this section is to find an orientable surface for each knot that has the knot as its boundary. This way we can assign a knot its genus. For this calculation we can use the formula in Theorem 6.16. But first we have to make the surface.

**Theorem 6.22** *For any knot $K$ we can construct an oriented polyhedral surface $S$, such that the boundary of $S$ is $K$.*

**Proof:** We simply construct the surface, known as the *Seifert surface.* We start by giving the knot an orientation, and taking a regular knot diagram. Pick any point on the knot, trace along the diagram in the direction of the orientation. When you encounter a crossing, change arcs once again following the orientation. Keep going until you end up retracing your path, when that happens start this again from an untraced point. Do this until the entire knot is traced.

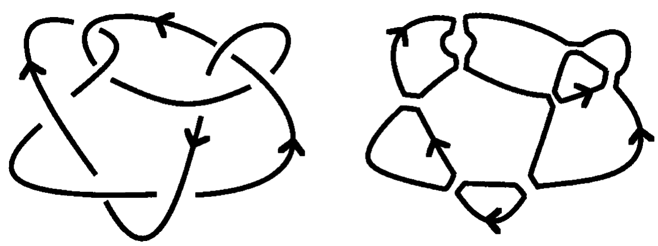


Figure 15: The result of such a procedure done on an oriented knot

Looking at the picture on the right, we can see that we made multiple shapes that are homeomorphic to a circle. (Take note that the big shape on the right is also one, and just has a smaller circle floating inside of it.) We call these, *Seifert circles*, and we will use them to construct a connected orientable surface.
Each circle is the boundary of a disk in the plane. We lift the circles that are nested in others, like the small one in the top right of figure 15. We connect the Seifert circles by attaching twisted bands at the points that correspond to the crossings in the diagram. The direction of the twist should correspond to the direction of the twist in the diagram.

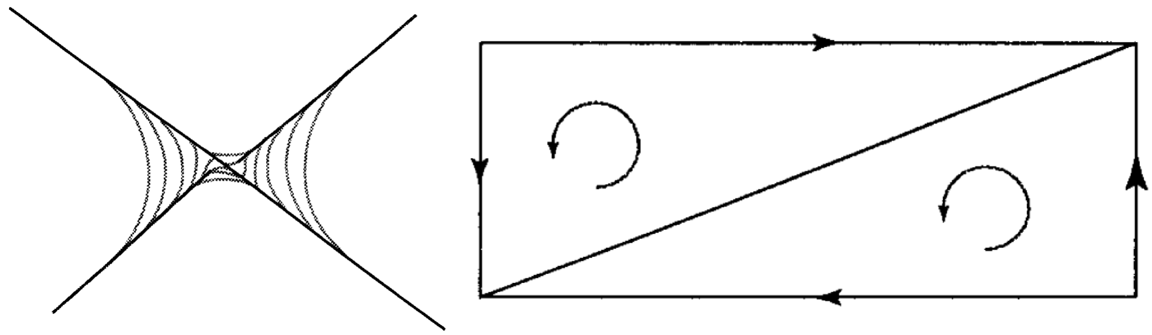


Figure 16: A twisted band, given smoothly and in polyhedral form as a twisted rectangle.

It is obvious that the knot is the boundary of this surface. This surface is called a *Seifert surface* of a knot. These are not unique, a knot can be the boundary of many Seifert surfaces, and also the boundary of surfaces that are not made in this way. A little more on this in Example 6.23.

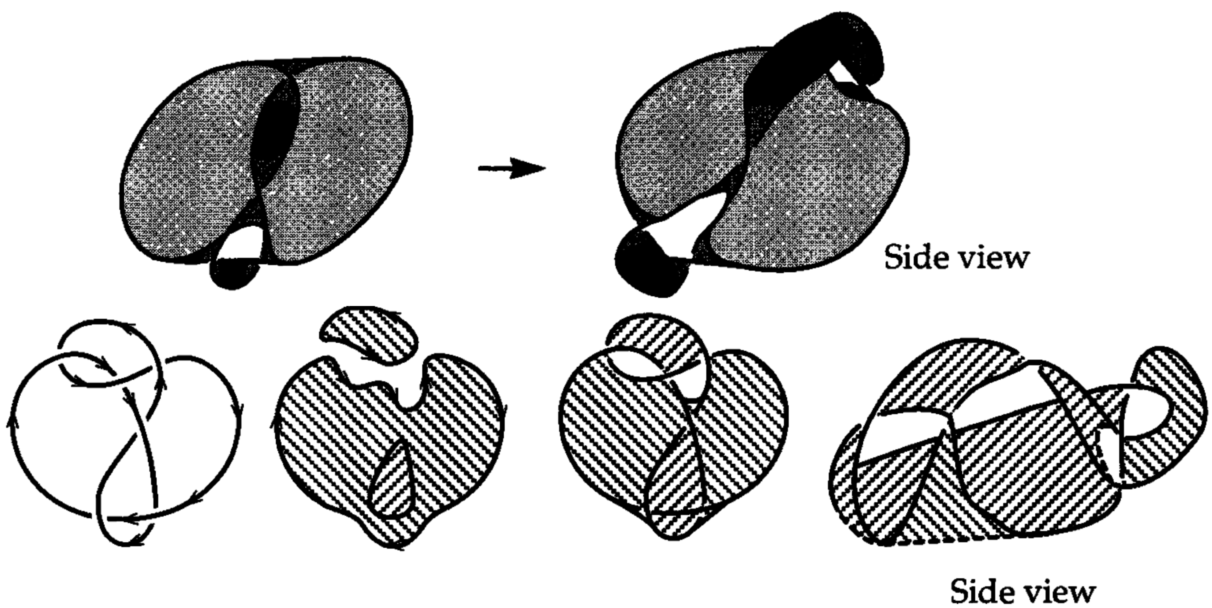



Figure 17: Some examples of Seifert surfaces

It is important that the Seifert surface is orientable because we want to use the formula from Theorem 6.16. So here is the proof of that:

Since we defined orientable surfaces using rotations on each triangle, each Seifert circle has an induced orientation given from the orientation of the knot. Any two Seifert circles that are connected with a twisted band will have different orientations when looking at from above. If they had the same orientation in the knot, they would have been the same Seifert circle. So the twisted bands do exactly what we want them to do. □

**Example 6.23** Some notes about the Seifert surface of the trefoil knot:

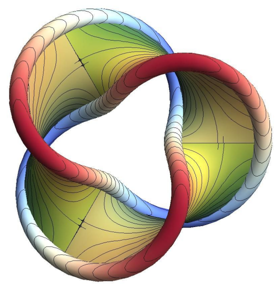


Figure 18: A surface of which its boundary is the trefoil knot. Take note that this surface is not a Seifert surface, because it is not orientable.

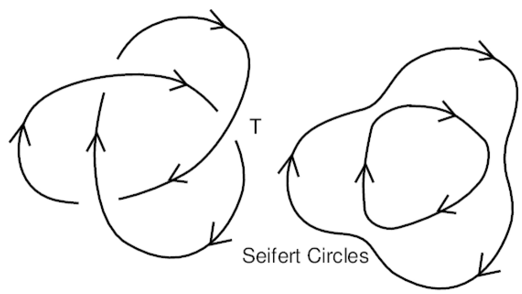



Figure 19: As you can see here, the Seifert surface would consist of two disks connected by three twisted bands. This surface is actually orientable, so we can compute the genus using our formula.

## 7 Back to knots

**Definition 7.1** The **genus of a knot** is the minimum possible genus of a Seifert surface for the knot. Notation: a knot $K$ with genus $n$ has $g(K) = n$.

Take note that calculating the genus of a knot can be really difficult as you would somehow have to show that the genus you found from your particular oriented knot diagram is minimal. Besides easy examples like the genus of the unknot being 0. We are mostly just going to use its existence to prove our desired results. Which brings us to the following theorem.

**Theorem 7.2** *If a knot $K = K_1 \# K_2$ then $g(K) = g(K_1) + g(K_2)$*

**Proof:** $g(K) \leq g(K_1) + g(K_2)$ is very easy. Take the Seifert surfaces with minimal genus for $K_1$ and $K_2$, then bind them together in such a way that it becomes the Seifert surface for their composition. Now Corollary 6.15 tells us that the genus of this surface is $g(K_1)+g(K_2)$. This only gives us an upper bound, because we may be able to make a Seifert surface of smaller genus with the full composition. This is not possible but requires proof.
$g(K) \geq g(K_1) + g(K_2)$ is as follows. Look at $K_1$ and $K_2$ in terms of boxed knots, choose the box of $K_1$, make it a sphere, we call it separating sphere S. Take a minimal genus Seifert Surface for $K$, the connected sum, and call it $F$. We do not know what this surface looks like, but we will show that there is a second surface $G$ with $g(F) = g(G)$, which can be described as the union of Seifert surfaces for $K_1$ and $K_2$, meeting in a single interval of their boundaries. And we once again use Corollary 6.15, which gives us the desired genus of $F$. The approach is to work with the intersection of $F$ and $S$. $F$ intersects $S$ in a collection of arcs and circles. (Initially, it could contain some isolated points. However, moving $F$ slightly will eliminate any such unexpected intersections.)
It should be clear that the only arc of intersection on $S$ runs from two points on $S$ that intersect $K$. Now we work with the circles of intersection, and we will use what is called *surgery* to eliminate them one by one.

*Surgery* goes as follows. $F$ Is a surface in 3-space and $D$ is a disk in 3-space. The interior of $D$ is disjoint from $F$, and the boundary of $D$ lies in the interior of $F$. And we construct a new surface, by removing a strip, or annulus, on $F$ along the circle where $F$ and $D$ meet. The new surface has two more boundary components than $F'$. To each of these boundary components, attach a disk which is parallel to $D$.

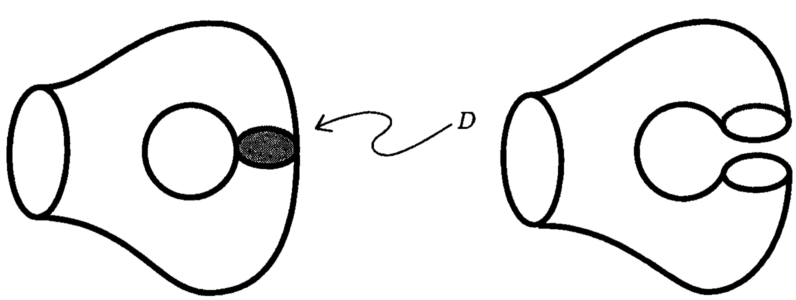



Figure 20: An intuitive idea of performing surgery on a surface.

Claim: If surgery is performed on $F$, which is a connected orientable surface, and it results in a connected surface $F'$. Then $g(F) = g(F') - 1$. If it instead results in two components $F'$ and $F''$, then $g(F) = g(F') + g(F'')$.
To prove this we just look at what happens to the Euler characteristic, often using Theorem 6.18. The Euler characteristic of an annulus is 0 so removing it has no effect on $\chi(F)$. The Euler characteristic of a disk is 1, so by the same theorem, the two disks add 2 to the Euler characteristic. We simply plug this into the formula of the genus, and $g(F') = g(F) - 1$. That is if we assume $F'$ is still connected. In the case that we split $F$ into $F'$ and $F''$, we look at each of their number of boundary components $B, B', B''$. Note that $B = B' + B''$ and thus:

$$\begin{aligned} g(F') + g(F'') &= \frac{1}{2}(2 - \chi(F') - B') + \frac{1}{2}(2 - \chi(F'') - B'') \\ &= \frac{1}{2}(4 - \chi(F') - \chi(F'') - B) \\ &= \frac{1}{2}(4 - (\chi(F) + 2) - B) \\ &= g(F) \end{aligned}$$

This proves the claim. We will now use this fact and the method of surgery to get our upper bound.

Consider an innermost circle of intersection. Meaning, one of the circles on $S$ that bounds a disk on $S$ containing no points of intersection of $F$ and $S$ in its interior. Surgery can now be performed on $F$ along this disk to construct a new surface bounded by $K$. If the new surface is connected, then it is a Seifert surface for $K$, which has lower genus than $F$ thanks to our claim. This is a contradiction with the assumption that $g(F)$ is minimal, so our surgery results in a disconnected surface. Remove the component that does not contain $K$. Using our claim the remaining surface has genus $\leq g(F)$, and the assumption that $g(F)$ is minimal implies it is an equality. This new surface will have fewer circles of intersection with $S$ since the circle along which the surgery was done is no longer on the surface.
Repeating this construction, a surface $G$ is our end result. Since it meets $S$ only in an arc, it is formed as the union of Seifert surfaces for $K_1$ and $K_2$ that intersect in a single arc, as desired. With this we have proven $g(K) \geq g(K_1) + g(K_2)$. □

**Remark 7.3** This kind of argument can be referred to to as a *cut-and-paste* argument, since we are cutting out portions and pasting in new pieces of the surface. It can also be referred to as an *innermost circle* argument. It is typical for proofs in knot theory and geometric topology.

We now use this to prove Proposition 5.5 and Theorem 5.8, and solidify our idea that the connected sum of knots is comparable to the integers with multiplication. Both having a lack of inverses and the elements being build from primes.

**Proposition 7.4** *Restated Proposition 5.5:*
*Let $K$ be an oriented knot that is non-trivial. No oriented knot $J$ exists such that $K\#J$ is unknotted.*

**Proof:** First note that any knot with genus 0 is unknotted, because its Seifert surface with minimal genus would just be a (twisted) disk. We can untwist the knot in the same way, and show that it is trivial.
Assume the knot $J$ does exist. Theorem 7.2 then gives us $g(K)+g(J) = 0$ thus $g(K) = -g(J)$. We also know that $g(K) \neq 0$ because K is non-trivial and thus has a Seifert surface with genus $\geq 0$. We have concluded that $J$ has a negative genus, contradiction with Proposition 6.17. □

**Theorem 7.5** *Restated Theorem 5.8:*
*Every non-trivial oriented knot can be decomposed as the connected sum of nontrivial prime knots. And if $K = K_1\#\ldots\#K_n$ and $K = K_1'\#\ldots\#K_m'$ where each $K_i$, $K_i'$ are non-trivial prime knots, then $n = m$ and there exists a bijective function $\sigma : \{1,\ldots,n\} \rightarrow \{1,\ldots,n\}$ with the following property: $K_i$ is equivalent to $K'_{\sigma(i)}$ for $i \in \{1,\ldots,n\}$.*

**Proof:** The existence of a prime decomposition follows easily from an inductive argument using the additivity of the knot genus. So we will start with that. We have shown that the genus of any orientable surface is an integer $\geq 0$, so we can do induction on the genus of a knot.
Take a knot $K$ with $g(K) = 1$, then K is not unknotted. $K$ Is actually prime because any composition $J\#L = K$ implies that $g(J) = 0 \wedge g(L) = 1$ or $g(J) = 1 \wedge g(L) = 0$ using the additivity of the knot genus. We can conclude that one of the two knots must be unknotted.
Now assume any knot $K'$ with $g(K') \leq p$ decomposes into a connected sum of nontrivial prime knots. Take a knot $K$ with $g(K) = p$, then there are two possibilities:
-If $K$ is prime, we are done.
-If $K$ is not prime, a decomposition exists such that $J\#L = K$ where $0 < g(J) < p \wedge 0 < g(L) < p$. Which follows from $g(J) + g(L) = g(K)$ and the non triviality of $J$ and $L$. We use our induction hypothesis to decompose $J = K_1\#\ldots\#K_i$ and $L = K_{i+1}\#\ldots\#K_j$ and from this we gain our decomposition of $K = K_1\#\ldots\#K_j$ where all $K_l$ are nontrivial prime knots.

The uniqueness of the decomposition, up to permutation, is proven similarly to the additivity of the knot genus. The proof was a bit harder to find, and too hard to do myself so it is based on what I found in a different book [3].
We will first describe a general concept for the construction of prime decompositions of a given knot $K$. Then we show that any two decompositions can be connected by a chain of 'elementary processes'.

**Definition 7.6** Take $S_j$, $1 \leq j \leq m$ a system of disjoint 2-spheres in 3-space. bounding $2m$ balls $B_i$, $1 \leq i \leq 2m$ in 3-space, where $B_j$, $B_{c(j)}$ are the two balls bounded by $S_j$. If $B_i$ contains the $s$ balls $B_{l(1)},\ldots,B_{l(s)}$ as proper subsets, $R_i = B_i - \bigcup_{q=1}^{s} \mathring{B}_{l(q)}$ is called the *domain* $R_i$. The *spheres* $S_j$ are said to be *decomposing* with respect to a knot $K$ in 3-space if the following conditions are met:
1) Each sphere $S_j$ meets $K$ in two points.
2) The arc $K \cap R_i$, oriented the same as $K$, and completed by simple arcs on the boundary of $R_i$ to represent a knot $K_i \subseteq R_i \subseteq B_i$, is prime. $K_i$ Is called the *factor of $K$ determined by* $B_i$. With $\mathcal{S} = \{(S_j, K) \mid 1\ leqj \leq m\}$ we denote a decomposing sphere system with respect to $K$. If $K$ is prime, $\mathcal{S} = \emptyset$.

It is clear that $K_i$ does not depend on the choice of the arcs on the boundary of $R_i$. We use the following Lemma to connect this definition with knot composition.

**Lemma 7.7** *If $\mathcal{S} = \{(S_j, K) \,|\, 1 \leq j \leq m\}$ is a decomposing system of spheres, then there are balls $B_i$, $1 \leq i \leq m+1$ determining prime knots such that $K = K_{\sigma(1)}\#\ldots\#K_{\sigma(m+1)}$ where $\sigma$ is a permutation.*

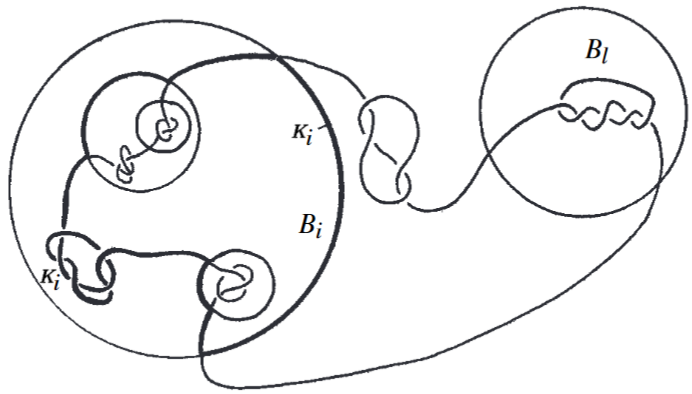



Figure 21: The definition and the idea of the lemma.

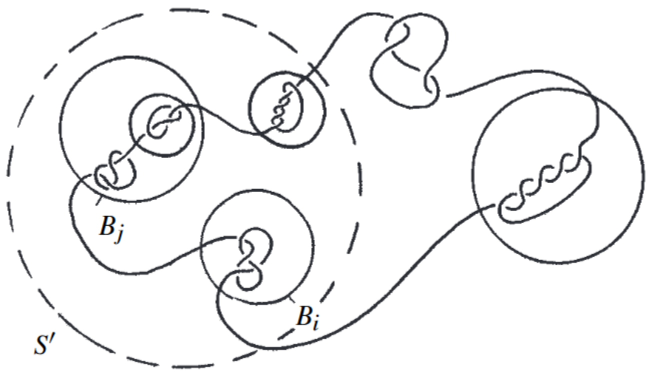
**Proof:** We use induction, $m = 0$ is trivial, and for $m = 1$ we refer to the definition of knot composition. Now take ball $B_l$ that does not contain any other $B_i$ and determine prime knot $K_l$. Replacing the knotted arc $B_l \cap K$ in $K$ by a simple arc on the boundary of $B_l$ defines a non-trivial knot $K'$ in 3-space. We apply the induction hypothesis to $\{(S_j, K') \,|\, 1 \leq j \leq m, \, j \neq l\}$ gives $K' = K_{\sigma(1)}\#\ldots\#K_{\sigma(m)}$. Now $K = K'\#K_l = K_{\sigma(1)}\#\ldots\#K_{\sigma(m+1)}$ where $\sigma(m+1) = l$. □

**Definition 7.8** Two decomposing systems of spheres $\mathcal{S} = \{(S_j, K)\}$ and $\mathcal{S}' = \{(S'_j, K)\}$, $1 \leq j \leq m$ are called if they define the same (unordered) (m+1) factor knots $K_{l(j)}$.

The following lemma is crucial in proving the uniqueness. We use this to pass over from one decomposing system to an equivalent one.

**Lemma 7.9** *Take a decomposing system of spheres $\mathcal{S} = \{(S_j, K) \mid 1 \leq j \leq m\}$ and an additional sphere $S'$, disjoint from all the $S_j$, that bounds the balls $B'$, $B''$. if $B_i$ whose boundary is $S_i$, is a maximal ball contained in $B'$. Meaning $B_i \subseteq B'$ and for any $j \neq i$ we have $\neg(B_i \subseteq B_j \subseteq B')$. And if $B'$ determines the knot $K_i$ relative to the spheres $\{S_j \,|\, 1 \leq j \leq m, \, j \neq i\} \cup \{S'\}$. Then this defines a decomposing system of spheres with respect to $K$ equivalent to $\mathcal{S}$.*

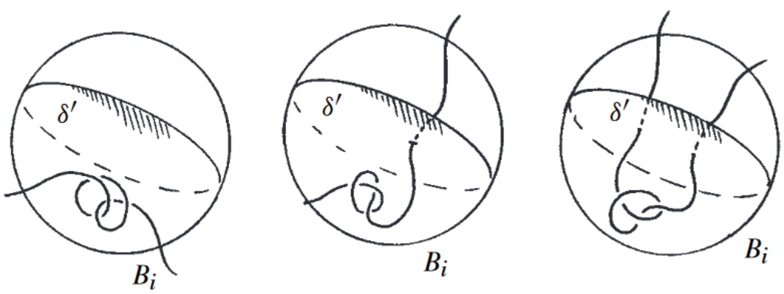
**Proof:** Take $K_j$ the knot determined by $B_j$ relative to $\mathcal{S}$, and assume $B_j \subseteq B'$. For $i \neq j$, $B_j$ determines the same knot $K_j$ relative to $\mathcal{S}'$ since no inclusion $B_i \subseteq B_j \subseteq B'$ exists per assumption. If there is a ball $B_l$ with $B_j \subseteq B_l \subseteq B_i$ then $B_{c(j)}$ determines $K_l$ relative to $\mathcal{S}$ and $\mathcal{S}'$. If no such $B_l$ exists, we have $K_{c(i)} = K_{c(j)}$ like in the following image:




Now $B_c(j)$ determines $K_i$ and $B''$ determines $K_{c(j)}$. So instead of $K_i, K_{c(i)}, K_j, K_{c(j)} = K_{c(i)}$ determined by $B_i, B_{c(i)}, B_j, B_{c(j)}$ in $\mathcal{S}$. We get $K_i, K_{c(i)}, K_j, K_i$ determined by $B', B'', B_j, B_{c(j)}$ in $\mathcal{S}'$. The case $B_j \subseteq B''$ is dealt with similarly. □

We now continue the proof of the main theorem. Which consists in verifying the assertion that any two decomposing systems $\mathcal{S} = \{(S_j, K) \mid 1 \le j \le m\}$ and $\mathcal{S}' = \{(S_j, K) \mid 1 \le j \le n\}$ with respect to the same knot are equivalent.

**Proof:** We do induction on $m+n$. The case that the sum is 0 is trivial. The spheres $S_j, S'_j$ can be assumed to be in general position relative to each other.
Supposed there is a ball $B_i \cap \mathcal{S}' = \emptyset$ not containing any other $B_j, B'_j$. Then by Lemma 7.9 some $S'_j$ can be replaced by $S_i$ and induction can be applies to $K \cap B_{c(j)}$.
If there is no such $B_i$ (or $B'_i$), choose an innermost curve $\lambda'$ of $S'_j \cap \mathcal{S}$ bounding a disk $\delta' \subseteq S'_j = \partial B'_j$ (where $\partial$ denotes the boundary) such that $B'_j$ contains no other ball $B_k$, $B'_l$. The knot $K$ meets $\delta'$ in at most 2 points. The disk $\delta'$ divides $B_i$ into two balls $B_i^1$, $B_i^2$, and in the first two cases of the following image, one of them determines the trivial knot or does not meet $K$ at all, and the other one determines the prime knot $K_i$ with respect to $\mathcal{S}$. Because otherwise $\delta'$ would effect a decomposition of $K_i$.




If $B_i^1$ determines $K_i$, replace $S_i$ by $\partial B_i^1$ or rather by a sphere $S'$ obtained from $\partial B_i^1$ by a small isotopy such that $\lambda'$ disappears and general position is restored. The new decomposing system is equivalent to the old one by lemma 7.9. If $K$ meets $\delta'$ in two points, the third case of the above image, we choose $\delta'' = S_j - \delta'$ instead of $\delta'$ if $\lambda'$ is the only intersection curve on $S'_j$. if not, there is another innermost curve $\lambda'' = S'_j \cap S_k$ on $S'_j$ bounding a disk $\delta'' \subseteq S'_j$. In both events the knot $K$ will not meet $\delta''$ and we are back to the first case in the image above. We obtain an innermost ball without intersections, and this proves the theorem using lemma 7.7. □

**Remark 7.10** During my presentation about this thesis, I received an interesting question from one of my professors, E. Cator. He asked if there are proven to be infinitely many prime knots. I had not thought of this myself funnily enough. My supervisor answered for me that there are. I considered adding this onto my thesis but decided not to due to time constraints with other courses.

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